

Exercises for 'Topics in complex analysis'

(17/09/2025)

H 2.1 (An example of propagation of convergence)

Let $D \subset \mathbb{C}$ be a bounded domain and denote its closure by \overline{D} . Let $f_n : \overline{D} \rightarrow \mathbb{C}$ be a sequence of continuous functions such that each f_n is holomorphic in D . Assume that the sequence f_n converges uniformly on the boundary ∂D . Show that the sequence f_n converges uniformly on the whole set \overline{D} to some $f : \overline{D} \rightarrow \mathbb{C}$.

Hint: Show that f_n is a Cauchy-sequence with respect to uniform convergence.

H 2.2 (On sequences of holomorphic functions)

a) Let $f_n : B_1(0) \rightarrow \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly to a holomorphic function $f : B_1(0) \rightarrow \mathbb{C}$. Does the sequence $f_n^{(n)}$ converge locally uniformly to some continuous function $g : B_1(0) \rightarrow \mathbb{C}$? Give a proof or find a counterexample.

b) Give an example of an open set $U \subset \mathbb{C}$ and a sequence of holomorphic functions $f_n : U \rightarrow \mathbb{C}$ that converges locally uniformly to some $f : U \rightarrow \mathbb{C}$ and such that each f_n has exactly one zero, while f has no zeros.

H 2.3 (Convergence of varying path-integrals)

Let $f_n : U \rightarrow \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly on an open set $U \subset \mathbb{C}$ to some $f : U \rightarrow \mathbb{C}$. Let further $\gamma_n : [0, 1] \rightarrow U$ be a sequence of C^1 -paths such that $\gamma_n \rightarrow \gamma$ and $\gamma_n' \rightarrow \gamma'$ uniformly on $[0, 1]$ for some C^1 -path $\gamma : [0, 1] \rightarrow U$. Show that

$$\lim_{n \rightarrow +\infty} \int_{\gamma_n} f_n(z) dz = \int_{\gamma} f(z) dz.$$

H 2.4 (Osgood's theorem)

Let $U \subset \mathbb{C}$ be open and $f_n : U \rightarrow \mathbb{C}$ be a sequence of holomorphic functions that converges pointwise to a function $f : U \rightarrow \mathbb{C}$. In this exercise we show that there exists an open, dense subset $U_0 \subset U$ such that the sequence f_n is locally uniformly bounded on U_0 . As we will see in the course, this actually implies the local uniform convergence of f_n to f on U_0 , and consequently that f is holomorphic on U_0 .

(i) Let U_0 be the set of points for which $(f_n)_{n \in \mathbb{N}}$ is uniformly bounded in a neighborhood, i.e.

$$U_0 := \{z \in U : \exists r > 0 \text{ s.t. } \sup_{n \in \mathbb{N}} \sup_{z' \in B_r(z)} |f_n(z')| < +\infty\}.$$

Show that U_0 is open.

(ii) Show that if U_0 is not dense in U , then there exists a ball $B_{r_0}(z_0) \subset U$ such that for all balls $B_{r'}(z') \subset B_{r_0}(z_0)$ the sequence f_n is not uniformly bounded on $B_{r'}(z')$.

(iii) Use (ii) to find a sequence of nested closed balls $\overline{B_{r_k}(z_k)} \subset B_{r_{k-1}}(z_{k-1})$ and a subsequence f_{n_k} such that $|f_{n_k}| \geq k$ on $\overline{B_{r_k}(z_k)}$.

- (iv) Recall that due to compactness there exists an element \bar{z} that belongs to all $\overline{B_{r_k}(z_k)}$. Derive a contradiction to the pointwise convergence of f_n and conclude the proof.